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Dilatations and factorizable equations

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Abstract

In this paper, we show that the dilatation operator in one dimension can be used to enlarge the Lie algebra generated by the raising and lowering operators for some classes of special functions. As a result, we are able to derive new recursion relations and addition formulae for these functions. Furthermore, we derive generalized ladder operators for these functions under coordinate stretching and translations.

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1. Introduction

The popularity of the factorization method in theoretical physics is due in part to the seminal paper of Infeld and Hull [1] and its close relation to the functions of mathematical physics. Since the publication of this paper, various extensions of the method to systems of differential equations appeared in the literature [2–5] and the Lie algebraic contents and implications of this method were discussed by various authors [6, 7]. Recently, however, there was renewed interest in the application of this method to shape invariant potentials [8] and to problems in supersymmetric quantum mechanics [9, 10]. A recent paper [9] provides a comprehensive bibliography to these applications.

Our objective in this paper is to show that there are still some Lie algebraic aspects of some classes of special functions (which are solutions to factorizable equations) that were not recognized in the literature. Thus, when the Lie algebraic aspects of these functions were considered in the literature they were limited to the Lie algebra generated by the raising and lowering operators. Here we show that in addition to these operators, the dilatation and the translation operators can be added to these Lie algebras for some families of factorizable equations. This is in spite of the fact that these operators are not in the universal enveloping algebra generated by the raising and lowering operators. (However, the dilatation operator did appear in related applications [11, 12].) To put this enlargement in more general context, we note that extremal projection operators (which are also outside the universal algebra of the raising and lowering operators) were found for various Lie algebras. These operators are used to resolve various classification problems related to the representations of these algebras [13, 14] and form an active research field.

As a consequence of the enlargement of the Lie algebra acting on these families of special function, we are able derive new recursion relations and 'addition formulae'. These were always considered as an important issue in the field of special functions and mathematical physics in general. They are used in many applications and have a considerable importance for various approximation schemes as is attested by the 'Bateman project' [15] and Truesdell monograph on the special functions [16]. In addition to this we derive in this paper raising and lowering operators for these classes of functions under coordinate stretching or translations. (That is the dependence on x is replaced by a dependence on $\mathrm{e}^{\alpha}x$ or $x+\alpha$). We believe that these operators did not appear in the literature before. (As usual this last statement cannot be considered as 'absolute').

The plan of the paper is as follows: In section 2, we discuss and derive some formulae for the action of the dilatation operator and its powers. In section 3, we show that this operator can be used to enlarge the Lie algebra generated by the ladder operators for some classes of special functions and discuss its action on these functions. We also obtain new recursion relations for these classes of special functions. In section 4, we use these facts to derive some addition formulae for the Hermite polynomial and Bessel functions under the dual action of translations and dilatations. In section 5, we derive new ladder operators for these classes of functions with dilated and translated variables. We end in section 6 with some conclusions.

2. Dilatation and translation operators in one dimension

The canonical differential realization of dilatation and translation operators in one dimension is given by

$$D = x\frac{\partial}{\partial x}, \qquad T = \frac{\partial}{\partial x}. \tag{2.1}$$

The commutation relation between these operators is

$$[D, T] = -T. \tag{2.2}$$

The action of the one parameter groups generated by these operators on a smooth function $f(x)$ is given by

$$\mathrm{e}^{\alpha D} f(x) = f(\mathrm{e}^{\alpha}x), \qquad \mathrm{e}^{\beta T} f(x) = f(x+\beta). \tag{2.3}$$

Furthermore by induction we have that

$$D^m T = (-1)^m T(I-D)^m \tag{2.4}$$

(where I is the identity operator). It then follows that

$$\mathrm{e}^{\alpha D} T = \mathrm{e}^{-\alpha} T\, \mathrm{e}^{\alpha D}. \tag{2.5}$$

We can now use equations (2.2), (2.4) to show that

$$D^m T^n = (-1)^m T(I-D)^m T^{n-1} \tag{2.6}$$

and hence

$$\mathrm{e}^{\alpha D} T^n = \mathrm{e}^{-n\alpha} T^n\, \mathrm{e}^{\alpha D}. \tag{2.7}$$

Finally we obtain

$$\mathrm{e}^{\alpha D}\, \mathrm{e}^{\beta T} = \mathrm{e}^{\beta\, \mathrm{e}^{-\alpha}}\, \mathrm{e}^{\beta T}\, \mathrm{e}^{\alpha D}. \tag{2.8}$$

Since $D = xT$ we can prove (by induction) that

$$D^k = \sum_{m=0}^{k} \gamma_m^k x^m T^m \tag{2.9}$$

where γ_m^k satisfy the recursive relation

$$\gamma_m^k = \gamma_{m-1}^{(k-1)} + m\gamma_m^{(k-1)}, \qquad 2 \leqslant m \leqslant k-1 \tag{2.10}$$

and

$$\gamma_1^k = \gamma_k^k = 1. \tag{2.11}$$

In fact we have

$$D^2 = xTxT = x^2T^2 + xT, \qquad D^3 = x^3T^3 + 3x^2T^2 + xT \tag{2.12}$$

which verifies equations (2.10), (2.11) for $k = 2, 3$. Assuming equation (2.9) is true for k we have for $k+1$

$$D^{k+1} = D(D^k) = xT\left(\sum_{m=0}^{k} \gamma_m^k x^m T^m\right) \tag{2.13}$$

and the relations (2.10), (2.11) now follow from the commutation relation $[T, x^n] = nx^{n-1}$. (The coefficients γ_m^k which are introduced here are actually the Stirling numbers of the second kind [19].)

In the notation of [1], a second-order differential equation in normal form

$$y''(\lambda, m, x) + r(x, m)y(\lambda, m, x) + \lambda y(\lambda, m, x) = 0 \tag{2.14}$$

is factorizable if one can find first-order (ladder) operators R_m, L_m in the form

$$R_m = k(m+1, x) - \frac{\mathrm{d}}{\mathrm{d}x}, \qquad L_m = k(m, x) + \frac{\mathrm{d}}{\mathrm{d}x} \tag{2.15}$$

so that the eigenfunctions of equation (2.14) satisfy

$$R_m y(\lambda, m, x) = \sqrt{\lambda - \Gamma(m+1)}\, y(\lambda, m+1, x) \tag{2.16}$$

and

$$L_m y(\lambda, m, x) = \sqrt{\lambda - \Gamma(m)}\, y(\lambda, m-1, x) \tag{2.17}$$

where Γ is some function of m only (*this is not the Euler* Γ *function*). In the following we suppress for brevity the dependence of $y(\lambda, m, x)$ on λ. We also observe that the dependence of R_m, L_m on m can be replaced by a differential operator (so that these operators become 'pure' differential operators acting on $L^2(R^2)$ as was done in [5]).

It was shown in the literature [5, 6] that the factorization method has a Lie algebraic content and the ladder operators with their commutator close a Lie algebra. This fact combined with the action of the corresponding one parameter groups of these operators has led to the derivation of various sum rules for some classes of special functions [5, 6, 9] and many other applications in theoretical physics [8].

In the following, we show that in some cases the Lie algebra generated by the ladder operators can be enlarged by the addition of the dilatation and translation operators and this leads to a richer structure for the corresponding eigenfunctions.

3. Lie algebras of some factorizable equations

In this section, we give some specific examples of factorizable equations for which the Lie algebra of the ladder operators can be enlarged to include dilatations and translations. We then use these operators to derive new 'higher order' recursion relations for these classes of special functions.

3.1. Hermite polynomials

Hermite polynomials $H_n(x)$ satisfy

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \tag{3.1}$$

where n is an integer. The ladder operators for these functions are

$$R = 2x - \frac{\mathrm{d}}{\mathrm{d}x}, \qquad L = \frac{\mathrm{d}}{\mathrm{d}x}. \tag{3.2}$$

Their action on $H_n(x)$ is given by

$$RH_n(x) = H_{n+1}(x), \qquad LH_n(x) = 2nH_{n-1}(x) \tag{3.3}$$

and their commutation relations yield

$$[R, L] = -2I. \tag{3.4}$$

We can now enlarge this Lie algebra by adding the 'exterior' dilatation operator D. It is straightforward to verify the following commutation relations:

$$[D, L] = -L, \qquad [D, R] = R + 2L. \tag{3.5}$$

To obtain the explicit action of the operator D on $H_n(x)$ we rewrite D as xL and hence

$$DH_n(x) = 2nxH_{n-1}(x). \tag{3.6}$$

Thus, the extended Lie algebra for the Hermite polynomials contains the four operators $\{R, L, D, I\}$.

For higher powers of D we have the recursion relation

$$D^k H_n(x) = 2nx(D+1)^{k-1}H_{n-1}(x), \qquad 1 \leqslant k \leqslant n. \tag{3.7}$$

However in this case $T = L$ and therefore

$$T^m H_n(x) = 2^m \frac{n!}{(n-m)!} H_{n-m}(x), \qquad 1 \leqslant m \leqslant n. \tag{3.8}$$

From equation (2.9) it then follows that

$$D^k H_n(x) = \sum_{m=0}^{k} 2^m \frac{n!}{(n-m)!} \gamma_m^k x^m H_{n-m}(x), \qquad 1 \leqslant k \leqslant n. \tag{3.9}$$

To demonstrate the applicability of these relations we use them to rederive the basic recursion relation for the Hermite polynomials. Thus using equation (3.3) we obtain

$$[D, R]H_n = (R + 2L)H_n = H_{n+1} + 4nH_{n-1} \tag{3.10}$$

and

$$D(R(H_n)) - R(D(H_n)) = 2(n+1)xH_n - (xR - 1)(2nH_{n-1}) = 2xH_n + 2nH_{n-1} \tag{3.11}$$

where we have used the commutation relation $[R, x] = -1$. Combining equations (3.10), (3.11) we obtain

$$H_{n+1} - 2xH_n + 2nH_{n-1} = 0. \tag{3.12}$$

This is the basic recursion relation for the Hermite polynomial. Its derivation here demonstrates that it is due to the properties of these functions under the action of the dilatation operator.

The recursion relations given by equations (3.8), (3.9) are new generalizations of this basic recursion relation for the Hermite polynomials.

3.2. Hermite functions

Hermite functions (or equivalently the eigenfunctions of the Harmonic oscillator) satisfy the differential equation

$$h_n''(x) - x^2 h_n'(x) + (2n+1)h_n(x) = 0 \tag{3.13}$$

where n is an integer. The ladder operators for these functions are

$$R = x - \frac{\mathrm{d}}{\mathrm{d}x}, \qquad L = x + \frac{\mathrm{d}}{\mathrm{d}x}. \tag{3.14}$$

These operators are multiplied sometimes by a constant whose value depends on the normalization chosen for $h_n(x)$. (Here we follow [1, 5] which normalize the L^2 norm of these functions to 1.)

The action of these operators on $h_n(x)$ is given by

$$Rh_n(x) = [(2(n+1)]^{1/2}h_{n+1}(x), \qquad Lh_n(x) = (2n)^{1/2}h_{n-1}(x) \tag{3.15}$$

with $h_0(x) = \mathrm{e}^{-x^2/2}$.

These ladder operators R, L satisfy the same commutation relations as in equation (3.4). It then follows that the corresponding Lie algebra can be enlarged as before by the addition of the dilatation operator and we have

$$[D, L] = R, \qquad [D, R] = L. \tag{3.16}$$

We also observe that $T = \frac{L-R}{2}$.

Since $D = x\frac{L-R}{2}$ we have the following recursive relation for $k \geqslant 1$:

$$D^k h_n(x) = \frac{x}{2}(D+1)^{k-1}[(2n)^{1/2}h_{n-1}(x) - ((2(n+1))^{1/2}h_{n+1}(x)]. \tag{3.17}$$

A more explicit formula for this action can be derived again by the use of equation (2.9) which leads to

$$D^k h_n(x) = \sum_{m=0}^{k} 2^{-m}\gamma_m^k x^m (R-L)^m h_n(x). \tag{3.18}$$

It is easy to compute these higher order recursion relations explicitly for low values of k. However, for arbitrary values of k the evaluation of these recursion relations requires the normal ordering of general monomials in the operators R, L which is not a straightforward task [17–19] (and references cited therein).

3.3. Bessel functions

The Bessel differential equation is

$$J_n''(r) + \frac{1}{r}J_n'(r) - \left(\frac{n^2}{r^2} - 1\right)J_n(r) = 0 \tag{3.19}$$

where n is an integer. The ladder operators for these functions are

$$R = \frac{n}{r} - \frac{\mathrm{d}}{\mathrm{d}r}, \qquad L = \frac{n}{r} + \frac{\mathrm{d}}{\mathrm{d}r}. \tag{3.20}$$

Strictly speaking these operators should be denoted by R_n and L_n. However, in the following we shall drop this subscript unless it is needed. It is possible, however, to obtain a differential expression independent of n for these operators as was done in [5]. The action of these operators on $J_n(r)$ is given by

$$RJ_n(r) = J_{n+1}(r), \qquad LJ_n(r) = J_{n-1}(r). \tag{3.21}$$

Hence they satisfy the commutation relations

$$[R, L] = 0. \tag{3.22}$$

We can enlarge this algebra by the addition of the dilatation operator $D = r\frac{\mathrm{d}}{\mathrm{d}r}$ and we obtain

$$[D, L] = -L, \qquad [D, R] = -R \tag{3.23}$$

(T is already in the algebra). To obtain the action of D on $J_n(r)$ we rewrite it as $D = r\frac{L_n - R_n}{2}$ (which is true for any n). This leads to the following recursive relation for $k \geqslant 1$:

$$D^k J_n(r) = \frac{r}{2}(D+1)^{k-1}[J_{n-1}(r) - J_{n+1}(r)]. \tag{3.24}$$

In this formula, we must re-express D in terms of R_n and L_n according the index of the Bessel function on which it acts. To derive a more explicit formula for this action we observe that $T = \frac{L_n - R_n}{2}$ for any n and use equation (3.22) to obtain

$$T^m J_n(r) = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k} J_{n+m-2k}(r). \tag{3.25}$$

Using equation (2.9) then yields

$$D^k J_n(r) = \sum_{m=0}^{k}\frac{1}{2^m}\gamma_m^k x^m \sum_{k=0}^{m}\binom{m}{k} J_{n+m-2k}(r). \tag{3.26}$$

These are again new higher order recursion relations for these functions.

As a simple application of the commutation relations between D and R_n, we derive the following recursion relation. Using

$$[D, R_n]J_n(r) = -R_n J_n(r) = -J_{n+1}(r) \tag{3.27}$$

and utilizing the relation

$$T = \frac{\mathrm{d}}{\mathrm{d}r} = \frac{L_n - R_n}{2} \tag{3.28}$$

for any n leads to

$$D R_n J_n(r) = \frac{r}{2}[J_n(r) - J_{n+2}(r)] \tag{3.29}$$

$$R_n D J_n(r) = \frac{R_n}{2}[r J_{n-1}(r) - r J_{n+1}(r)]. \tag{3.30}$$

Simplifying equation (3.30) by using the commutation relation $[R_n, r] = -1$ and combining it with equations (3.27)–(3.29) yields the recursion relation

$$(n+1)J_{n+1} - (n-1)J_{n-1} + \frac{r}{2}(J_{n-2} - J_{n+2}) = 0. \tag{3.31}$$

3.4. Associated Laguerre's polynomials

These polynomials $L_{n,m}(t)$ satisfy the differential equation

$$z''(t) + (m+1-t)z'(t) + nz(t) = 0 \tag{3.32}$$

where n, m are integers. To derive ladder operators for these functions we rewrite (as first step) this equation in self-adjoint form

$$\frac{\mathrm{d}}{\mathrm{d}t}[t^{m+1}\,\mathrm{e}^{-t}z'(t)] + nt^m\,\mathrm{e}^{-t}z(t) = 0. \tag{3.33}$$

Introducing

$$x = 2t^{1/2}, \qquad y = t^{m/2+1/4}\,\mathrm{e}^{-t/2}z(t) \tag{3.34}$$

equation (3.33) is transformed to

$$y''(x) + \left[-\frac{m^2 - 1/4}{x^2} + \frac{(m+1)}{2} - \frac{x^2}{16} + n\right] y(x) = 0. \tag{3.35}$$

This is a class C factorizable equation [1] and the ladder operators for $y_{n,m}$ are

$$R = \left[\frac{m+1/2}{x} + \frac{x}{4}\right] - \frac{\mathrm{d}}{\mathrm{d}x}, \qquad L = \left[\frac{m-1/2}{x} + \frac{x}{4}\right] + \frac{\mathrm{d}}{\mathrm{d}x} \tag{3.36}$$

and their action on $y_{n,m}$ is given by

$$Ry_{n,m} = \sqrt{n-(m+1)}\,y_{n,m+1}, \qquad Ly_{n,m} = \sqrt{n-m}\,y_{n,m-1}. \tag{3.37}$$

Observe that these are first-order operators in contrast to the second-order raising and lowering operators that were found in [9] for these functions. These operators satisfy $[R, L] = I$. Although D does not close a Lie algebra with R, L it satisfies $[D, R - L] = -(R - L)$. (Observe that $R - L$ is independent of m). Moreover, since this commutation relation is formally the same as that between D and T (see equation (2.2)) we infer that

$$\mathrm{e}^{\alpha D}\,\mathrm{e}^{\beta(R-L)} = \mathrm{e}^{\beta\,\mathrm{e}^{-\alpha}}\,\mathrm{e}^{\beta(R-L)}\,\mathrm{e}^{\alpha D}. \tag{3.38}$$

Higher order recursion relations for the $y_{n,m}$ can be derived by considering the n successive applications of D to the commutation relation $[D, R - L] = -(R - L)$ which leads to $[D, [\ldots, [D, R - L]]\ldots] = (-1)^n(R - L)$. In particular the commutation relation $[D, R - L] = -(R - L)$ leads to

$$\begin{aligned} &x\{y''_{n,m} + \sqrt{n-(m+1)}\,y'_{n,m+1} - \sqrt{n-m}\,y'_{n,m-1}\} \\ &\qquad = \sqrt{n-m}\,y_{n,m-1} - \sqrt{n-(m+1)}\,y_{n,m+1} - y_{n,m}. \end{aligned} \tag{3.39}$$

We can simplify this recursion relation further by replacing y'' by its equivalent using equation (3.35).

4. Addition formulae

In this section, we present some additional applications to the action of the dilatation and translation operators on the families of the special functions which were discussed above. In particular, we present examples how the Lie group action of these operators can be used to derive addition formulae under the combined action of translations and dilatations. Similar results can be obtained for the Hermite functions and the associated Laguerre polynomials. To some extent the results in this section can be viewed as special applications of the formulae derived in [19–22] for the general action of operators in the form $\exp\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x} + g(x)\right)$ (where $f(x)$, $g(x)$ are some general functions).

4.1. Hermite polynomials

To derive addition formulae for the Hermite polynomials, we apply the operator $\mathrm{e}^{\alpha D}$ by itself or in combination with $\mathrm{e}^{\beta L}$, $\mathrm{e}^{\beta R}$. Thus

$$\mathrm{e}^{\alpha D} H_n(x) = H_n(\mathrm{e}^{\alpha}x) = \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} D^m H_n(x) \tag{4.1}$$

where $D^m H_n$ has been evaluated in equation (3.9). Similarly we can apply the operator $e^{\alpha D} e^{\beta L}$ to obtain

$$e^{\alpha D} e^{\beta L} H_n(x) = H_n(e^{\alpha} x + \beta) = \sum_{m=0}^{\infty} \frac{\beta^m}{m!} \sum_{k=0}^{m} 2^k \binom{m}{k} (e^{\alpha D} H_{n-k}(x)) \tag{4.2}$$

where the action of $e^{\alpha D}$ on each $H_j(x)$ is given by equation (4.1). As these formulae involve infinite summation they can be used in various approximation schemes.

4.2. Bessel functions

On the Lie group level, we can obtain an expression for $e^{\alpha D} J_n(r)$ which is similar to equation (4.1) which utilizes the recursion relation (3.24) or equation (3.26). For the dual action of dilatation and translations we have

$$e^{\alpha D} e^{\beta T} J_n(r) = J_n(e^{\alpha} r + \beta) \tag{4.3}$$

to obtain using equation (3.25)

$$J_n(e^{\alpha} r + \beta) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^j \beta^m}{2^m m! j!} \sum_{k=0}^{m} \binom{m}{k} (D^j J_{n+m-2k}(r)) \tag{4.4}$$

where $D^j J_{n+m-2k}(r)$ can be evaluated using equation (3.26).

5. Generalized ladder operators

In this section, we derive generalized ladder operators for the classes of special functions which were discussed previously. These operators act on these functions when their dependence on x is replaced by $e^{\alpha} x$ or $x + \beta$.

To find these operators, we derive first formulae which relate the action $D^n R$ to that $R D^n$ (and similar formulae involving L, T). We then use these to relate the group action of D, T to those of the regular ladder operators R, L on these functions (i.e. compute the relation between the action $e^{\alpha D} R$ and $R e^{\beta D}$ etc).

5.1. Hermite polynomials

In this case the Lowering operator L is also the translation operator. Hence we need to compute only the formulae involving R and D.

A short calculation using equation (3.5) shows that the following holds:

1. When n is odd

$$D^n R = R(D+1)^n + 2L \sum_{m=0}^{[n/2]} \binom{n}{2m} D^{2m}. \tag{5.1}$$

2. When n is even

$$D^n R = R(D+1)^n + 2L \sum_{m=0}^{n/2-1} \binom{n}{2m+1} D^{2m+1}. \tag{5.2}$$

We then infer that

$$e^{\alpha D} R = [e^{\alpha} R + 2L \sin h\alpha]\, e^{\alpha D}. \tag{5.3}$$

Applying this relation to $H_n(x)$ yields

$$H_{n+1}(e^{\alpha} x) = [e^{\alpha} R + 2L \sin h\alpha] H_n(e^{\alpha} x) \tag{5.4}$$

i.e. the operator $e^{\alpha} R + 2L \sin h\alpha$ is the raising operator for the functions $H_n(e^{\alpha} x)$.

Using equations (2.7), (3.2) we find that for the lowering operator L

$$L^m H_n(e^{\alpha} x) = e^{m\alpha} 2^m m! \binom{n}{m} H_{n-m}(e^{\alpha} x), \qquad m \leqslant n \tag{5.5}$$

i.e. the generalized lowering operator is $e^{-\alpha} L$.

For the translation operator ($=L$) we have from equation (3.4)

$$L^n R = R L^n + 2n L^{n-1}. \tag{5.6}$$

Hence

$$e^{\alpha L} R = [R + 2\alpha]\, e^{\alpha L}. \tag{5.7}$$

By applying this relation to $H_n(x)$ we obtain

$$H_{n+1}(x + \alpha) = [R + 2\alpha] H_n(x + \alpha). \tag{5.8}$$

We observe that we can now combine the action of the translation and dilatation operators with R (or L) to obtain ladder operators acting on Hermite polynomials with coordinate stretching and translations. Thus

$$e^{\beta L}\, e^{\alpha D} R = [e^{\alpha} R + 2\beta\, e^{\alpha} + 2L \sin h(\alpha)]\, e^{\beta L}\, e^{\alpha D}. \tag{5.9}$$

This leads to

$$H_{n+1}(e^{\alpha} x + \beta) = [e^{\alpha} R + 2\beta\, e^{\alpha} + 2L \sin h(\alpha)] H_n(e^{\alpha} x + \beta). \tag{5.10}$$

(Similar observations hold for the Hermite and Bessel functions which are treated next.)

5.2. Hermite functions

We follow the same procedure outlined in the previous sub-section to derive generalized raising and lowering operators for these functions. It is easy to show that the following holds:

1. When n is odd

$$D^n R = R \sum_{m=0}^{[n/2]} \binom{n}{2m+1} D^{2m+1} + L \sum_{m=0}^{[n/2]} \binom{n}{2m} D^{2m}. \tag{5.11}$$

2. When n is even

$$D^n R = R \sum_{m=0}^{n/2} \binom{n}{2m} D^{2m} + L \sum_{m=0}^{n/2} \binom{n}{2m-1} D^{2m-1}. \tag{5.12}$$

From these formulae we infer that

$$e^{\alpha D} R = [\cos h(\alpha) R + \sin h(\alpha) L]\, e^{\alpha D}. \tag{5.13}$$

Applying this relation to $h_n(x)$ yields

$$[2(n+1)]^{1/2} h_{n+1}(e^{\alpha} x) = [\cos h(\alpha) R + \sin h(\alpha) L] h_n(e^{\alpha} x) \tag{5.14}$$

i.e. the operator $\cos h(\alpha) R + \sin h(\alpha) L$ is the raising operator for the functions $h_n(e^{\alpha} x)$.

Since the commutation relations (3.16) are symmetric in L and R it follows that the lowering operator for these function is

$$[2n]^{1/2}h_{n-1}(\mathrm{e}^{\alpha}x) = [\sin h(\alpha)R + \cos h(\alpha)L]h_n(\mathrm{e}^{\alpha}x). \tag{5.15}$$

For the Hermite functions, the translation operator satisfies $T = \frac{L-R}{2}$. Hence it is straightforward to show that

$$\mathrm{e}^{\alpha T}R = (R+\alpha)\,\mathrm{e}^{\alpha T}, \qquad \mathrm{e}^{\alpha T}L = (L+\alpha)\,\mathrm{e}^{\alpha T}. \tag{5.16}$$

Hence

$$[2(n+1)]^{1/2}h_{n+1}(x+\alpha) = (R+\alpha)h_n(x+\alpha), \quad [2n]^{1/2}h_{n-1}(x+\alpha) = (L+\alpha)h_n(x+\alpha). \tag{5.17}$$

5.3. *Bessel functions*

It is easy to show using equation (3.23) that

$$D^nR = R(D-1)^n, \qquad D^nL = L(D-1)^n \tag{5.18}$$

hence

$$\mathrm{e}^{\alpha D}R = \mathrm{e}^{-\alpha}R\,\mathrm{e}^{\alpha D}, \qquad \mathrm{e}^{\alpha D}L = \mathrm{e}^{-\alpha}L\,\mathrm{e}^{\alpha D}. \tag{5.19}$$

This leads to the following raising and lowering operators for these functions,

$$J_{n+1}(\mathrm{e}^{\alpha}r) = \mathrm{e}^{-\alpha}RJ_n(\mathrm{e}^{\alpha}r), \qquad J_{n-1}(\mathrm{e}^{\alpha}r) = \mathrm{e}^{-\alpha}LJ_n(\mathrm{e}^{\alpha}r). \tag{5.20}$$

Since the translation operator $T = \frac{L-R}{2}$ commutes with R, L we have

$$\mathrm{e}^{\alpha T}R = R\,\mathrm{e}^{\alpha T}, \qquad \mathrm{e}^{\alpha T}L = L\,\mathrm{e}^{\alpha T} \tag{5.21}$$

and therefore

$$J_{n+1}(r+\alpha) = RJ_n(r+\alpha), \qquad J_{n-1}(r+\alpha) = LJ_n(r+\alpha). \tag{5.22}$$

5.4. *Associated Laguerre polynomials*

Since D and the raising–lowering operators for the functions $y_{n,m}$ do not close a Lie algebra, we can derive only partial results in this case using the operator $R-L$. Since $[D, R-L] = -(R-L)$ simple calculation yields

$$\mathrm{e}^{\alpha D}(R-L) = \mathrm{e}^{-\alpha}(R-L)\,\mathrm{e}^{\alpha D}. \tag{5.23}$$

This implies

$$\sqrt{n-(m+1)}\,y_{n,m+1}(\mathrm{e}^{\alpha}x) - \sqrt{n-m-1}\,y_{n,m-1}(\mathrm{e}^{\alpha}x) = \mathrm{e}^{-\alpha}(R-L)y_{n,m}(\mathrm{e}^{\alpha}x). \tag{5.24}$$

6. Summary and conclusions

The dilatation operator plays a major role in many physical applications (e.g. quantum field theory). The fact that it can be used to enlarge the Lie algebra for some classes of factorizable equations is important in view of the key role that this method plays in various physical applications. In this short paper, we addressed only some of the basic mathematical applications of this enlargement of the Lie algebra and derived generalized ladder operators for some classes of special functions under coordinate stretching and translations.

From another point of view we observe that classical orthogonal families of functions were derived from solutions to differential equations. However, in the last 20 years wavelets which are classes of orthogonal functions with compact support appeared in the literature without any 'apparent link' to solutions of differential equations. These orthogonal classes of functions are generated by finite dilatations and translations of a 'mother wavelet function'. As shown in this paper solutions to some classes of factorizable equations have well-defined properties with respect to these operators. This result establishes a link (albeit a weak and indirect one) between these families of orthogonal functions (namely special functions and wavelets).

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